Mathematical Model for the Dynamic Behavior of the Demographic Transition

Roger D. Jones[1,2,*]
1 Roger D. Jones, Research Fellow, Center for Complex Systems and Enterprises, Stevens Institute of Technology, Hoboken, New Jersey, USA
2 Roger D. Jones, X-Center Network, San Jose, California, USA
∗ E-mail: Corresponding jones@xevents.com

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Abstract

A mathematical model (Core Model) is presented that describes the gross dynamic behavior of the demographic transition—falling death rates lead to population increase, temporarily rising birth rates, temporarily increased population growth, decreased fertility, aging of the population, and finally a steady population size higher than the initial population. Core Model captures these features. The model is based on three fundamental observations 1) people are born, 2) people die, some at a young age, the rest at an old age, and 3) people give birth more often when conditions are favorable than not favorable. In addition to boundary conditions, Core Model has one free parameter, which is associated with the rate at which fertility adapts to changing conditions. Core Model predicts that aging populations are a natural consequence of the dynamics and the speed at which the population fertility adapts to changing death rates. The model captures the qualitative features of actual country demographic dynamics with European countries making up most of the post-transition populations and Sub-Saharan Africa making up those countries just entering the transition.

Introduction

The human population is in the midst of an extreme social event, the *demographic transition*.[Rowland, 2003, pp.16-24][Thompson, 1929] [Notestein, 1945] [Kirk, 1996] [Dyson, 2010]: a trend from high birth and death rates to low birth and death rates with a concurrent increase in population. Population growth accompanies the transition because birth rates during the transition drop more slowly than death rates. Dyson noted that the transition is often accompanied by urbanization and aging of the population as well as possibly an improvement in economic conditions.[Dyson, 2010, and references therein] The transition is precipitated by a decline in mortality, which has been attributed to improved health and living conditions.[Dyson, 2010, p. 98] While the earlier transitions of Western European countries took more than 100 years to complete, the transitions that are currently occurring seem to be proceeding at the much faster pace of a few decades.[Dyson, 2010, p. 15] When compared with historical timescales of a few thousand years, this transition is almost instantaneous.

Much of the demographic transition process has been understood since the middle of the twentieth century. Since that time there has been extensive international experience on the evolution of the transition. The goal of this paper is to collect the understandings of the demographic transition and translate and organize them into a simple mathematical model. Presented in this paper is a closed, self-consistent mathematical model of the dynamics of the demographic transition that presents a simple framework for capturing the key causal relationships in the transition and makes testable predictions of global demographic changes. The model emulates demographic dynamics and relates measurable demographic variables in the data set to each other without reference to causal variables not included in the data. For example, this model does not directly include economic variables such as unemployment rate, or median income that likely affect demographic variables. In other words, all the information that is needed to build the self-consistent model is assumed to be embodied in the set of measurable demographic data. It will be seen that the model describes the dynamics of the shift between one equilibrium state of the population and another.

'Model' may have different meanings in different scientific contexts, so it is important to clarify what is meant by the term. A model may be 1) a verbal description of a process as in Darwin's theory of evolution by natural selection,[Darwin, 1999] 2) a statistical fit to data[Gourieroux and Monfort, 1997] as in least-squares regression, or 3) a hypothetical set of rules or mathematical equations that define key properties and causal relationships in the system such as Einstein's theory of Special Relativity.[Einstein, 1961]

The key requirement of each of these three types of model is that they must make at least one prediction that is not obvious from a straightforward observation of the data under study. For example, Einstein postulated that the speed of light c was constant in all frames of reference. One of several implications of this hypothesis was that mass m could be converted into energy E according to Einstein's famous formula $E = mc^2$. This non-obvious prediction was verified experimentally, thus lending credence to the speed-of-light hypothesis and the consequences of the supposition.

The model discussed here is described by definition 3): a rule-based set of hypothetical equations describing population dynamics associated with the demographic transition. We will describe the model and will identify non-obvious predictions suggested by the model. The postulated rules are designed to capture the essence of population dynamics on timescales longer than an expected lifetime. No attempt will be made to accurately model data on timescales shorter than a lifetime. While not unknown in demography, [Hilderink, 2000] [Jos Timmermans, 2008][Burch, 2003a][Burch, 2003b] the model-based approach adopted in this paper is more common in the natural sciences.

Because of the simplicity of the basic assumptions, the limited number of variables, and the simplicity of the mathematics, the demographic model in this paper is called Core Model. Core Model is based on three basic observations:

1. *People are born.*
2. *People die.* Some die in childhood, the rest at a later age.
3. *People give birth.* If there are so many children in a family that the family is in economic and social distress, the number of children women have tends to decrease. If there are very few children in the family, then biological, social, and economic pressures tend to increase the number of children in the family.

There are other minor assumptions, but these three observations capture the essence of the model.

Core Model is based on the following measurable demographic variables that describe the demographic transition:

$T^\dagger$ Median age of the population

T_e Average life span of both males and females in the population

d Crude death rate per capita. Crude rates exclude migration.

f Total fertility (births) per person (the replacement rate is f = 1)

b Crude birth rate per capita

N Number of people in the population

y Childhood death rate per capita. This is the per capita death rate for those people who die before they procreate.

Economic, social, cultural, biological, and political variables do not explicitly appear in the model, but are captured and considered causal factors for the demographic variables. For example, childhood death rate, y, is affected by improvements in public health and advances in medical technology and pharmacology—both economic factors. Moreover, the Core-Model demographic variables do not specifically address other demographic factors, such as migration or urbanization, which may affect these variables. The result is a closed, self-consistent model of the demographic transition that is based on demographic variables. Economic and other drivers are implicit in the demographic variables. In this regard, Easterlin [2013] provides a compelling cautionary tale on the direct use of external variables as causal agents.

Temporal age details are also not addressed in Core Model. The actual age that early death occurs is not considered. The model assumes that the age of early death is only important in that it occurs before the individual has the opportunity to procreate. The actual age of procreation is also not considered in Core Model. Despite this simplicity, Core Model is powerful. It is able to characterize a country's progress through the demographic transition, to provide insight into aging populations of developed countries, and to predict future population levels.

The first and most basic prediction of the Core Model is that demographic data appear to obey an Equation of State (EOS), an equation that relates groups of demographic variables to each other. Essential equations and behavior of the Core Model are presented first. The mathematical analysis and derivation details of the Core Model will be discussed in the Derivation Section.

Predictions of the Core Model

The consequences of Core Model are

1. Core Model captures the signature of the demographic transition–falling death rates lead to population increase, temporaritly rising birth rates, temporarily increased population growth, decreased fertility, aging of the population, and finally a steady population size higher than the initial population.

2. All countries seem to be in some stage of the demographic transition. European countries are the most mature, while Sub-Saharan African countries have the least mature dynamics. The model predicts that the fertility in aging countries such as Japan and the European countries will increase toward an equilibrium. Countries early in the transition such as the Sub-Saharan countries will become younger before the fertility starts to drop and the populations start to age. Other countries in the midst of transition will experience decreasing fertility.

3. There are two dimensionless ratios that define the dynamics of the population, one associated with how fast the population is growing, and the other associated with the age distribution in the population.

4. The dynamical states of the population are clustered around the curves generated by two equations known as the Stationary Manifold and the Equation of State. The two curves are attractors for the dynamical states.

5. Aging is a consequence of the speed with which the death rate has dropped and on how responsiveness of the fertility to non-uniform age distributions.

The presentation of results will begin with a discussion of the Equation of State.

Equation of State

The first prediction obtained from Core Model is that, *for populations that have low numbers of childhood deaths*, the demographic variables obey the *equation of state* (EOS)

$$\begin{aligned} \frac{b}{fd} &= \frac{T_e}{4T^\dagger - T_e} \quad \text{for } 2T^\dagger < T_e \quad \text{younger population} \\ &= \frac{3T_e - 4T^\dagger}{T_e} \quad \text{for } 2T^\dagger \geq T_e \quad \text{older population.} \end{aligned} \tag{1}$$

The EOS, Equation 1, follows from Equations 32 and 44 in the Derivation section of this paper. The EOS simply states that the ratio b/fd is a function of the life expectancy T_e and the median age $T^\dagger$. When $2T^\dagger < T_e$, then the ratio is given by the upper equation. When the converse is true, the ratio is equal to the function of T_e and $T^\dagger$ on the lower line. When $T_e = T^\dagger$, both the upper functions and the lower functions as well as the ratio are equal to one.

The ratio b/fd is composed of the vital demographic variables. Therefore, here it is given the name *vital ratio* represented by the symbol v.

$$v = \frac{b}{fd} \tag{2}$$

The vital ratio, which contains birth rates, death rates, and fertility rates, is one measure of the net increase or decrease in the population size.

Another important ratio is

$$a = \frac{T_e}{2T^\dagger}. \tag{3}$$

Here, a is called the *age ratio.* It is a measure of the age distribution in the population. If half the expected lifetime is greater than the median age ($a > 1$), then the population has more young people than old people. It is a young population. If half the expected lifetime is less than the median age ($a \leq 1$), then there are more old people than young people in the population. The population is older. High (low) median age with respect to the expected lifetime implies an older (younger) population.

The EOS can be written in terms of the ratios as

$$\begin{aligned} v &= \frac{a}{2-a} \quad \text{for } a > 1 \quad \text{younger population} \\ &= \frac{3a-2}{a} \quad \text{for } a \leq 1 \quad \text{older population.} \end{aligned} \tag{4}$$

Equation 4 simply states that, *for populations with no childhood deaths*, the growth or decline in the population size as represented by the vital ratio, v, depends only on the age distribution of the population. This agrees with intuition. If a population is aging ($a \leq 1$), then there are fewer young people giving birth and more old people dying. This lowers the vital ratio, ($v = b/fd$), for a given fertility, f. If the population is young ($a > 1$), there are more young women giving birth and fewer old people to die, thus increasing the vital ratio for a given fertility rate. This is stated mathematically in Equation 4.

The EOS is plotted in Figure 1. Note that the EOS constrains the age distribution in Core Model to lie in the interval $a \in (2/3, 2)$ for populations with *childhood death rate equal to zero.*

Mathematically, the EOS is a *manifold.* It relates one group of variables, b, d, and f, to another group, T_e and $T^\dagger$. The EOS defines a four-dimensional surface in the five-dimensional space of the five variables b, d, f, T_e and $T^\dagger$. The dynamics of Core Model is constrained to lie on the EOS manifold. This can be illustrated with a simple example. Consider the surface of the earth. It is a two-dimensional manifold in three-dimensional space. Except for the occasional space probe, humans are constrained to exist and move on or near the two-dimensional manifold that is the earth's surface. The EOS is a generalization of this concept to a four-dimensional manifold in a five-dimensional space. The dynamics of the core model. like human dynamics, is constrained to lie on or near its manifold. Another way of saying this is that the EOS manifold, like the earthly manifold, is an *attractor*. If the vital and age ratios are far from the EOS manifold, they are drawn toward the manifold much like objects are drawn to the earth's surface by gravity. The five variables b, d, f, T_e and $T^\dagger$ adjust such that the ratios v and a satisfy the EOS.

The observation that the variables are attracted to the EOS is a consequence of a simple assumption in Core Model regarding fertility. This assumption will be discussed at length in the Model Dynamics section. The assumption is that fertility adjusts to optimize the age distribution of the population. If the population is old, then the fertility increases to increase the number of young people, thus making the age distribution more uniform. If the population is young, then the fertility decreases in order to decrease the number of young people, once again making the age distribution more uniform. The fertility assumption is simply a restatement of Observation 3 in the Introduction.

One might argue, at this stage, that Core Model ignores much that is important in human dynamics such as economic, social, cultural, and political drivers. The gross features of the demographic transition described by Core Model, however, follow from the three basic observations 1) people are born, 2) people die, and 3) people adjust their birth rates depending on their well-being. The value of this approach can be illustrated with a simple physical analogy. Consider a pot of water. The water is in its liquid form—not ice and not steam. The state of the water, at least for simple purposes, is known if we know the pressure in the room, the temperature in the room, and the density of liquid water. These three variables are known as the state variables for water. Of course, there is much detail that has been left out. We have not considered that water is composed of tiny triangular-shaped molecules that have an electrical dipole moment, for example. All of this detail is contained in the equation of state for water. The equation of state is an equation that relates the three state variables. In other words, the equation of state allows, with minor algebraic manipulation, one to calculate any one of the state variables given that the other two are known. For instance, if the pressure and temperature of the room are known, then one can use the equation of state to calculate the density of the water. Now imagine that we hold the pressure in the room constant and then lower the temperature. At some point, when we lower the temperature, the water will freeze—it changes from liquid to solid. The equation of state predicts the temperature at which this happens. Also, when the water freezes, the density of the solid can be calculated from the equation of state. In the case of water, the solid state is less dense than the liquid state, which is why ice floats in liquid water. If we are interested in simple questions like how cold do we need to make liquid water in order to create ice for our drinks, then we do not need to know the details of the molecular properties of water. Moreover, the properties of ice are independent of the details of the coolant used to make the ice. It could be from a freezer or from natural atmospheric cooling, for instance. It is only necessary to know the state variables and the equation of state to answer many questions about the bulk properties of water.

The EOS for Core Model has a role in demographics similar to the role of the equation of state for water in thermodynamics. There are five state variables, b, d, f, T_e, and $T^\dagger$. The EOS relates these five variables such that if any four are known, the fifth can be calculated from the EOS with simple algebraic manipulation. The core model predicts that, if the childhood death rate is zero, the state variables are related to each other by Equation 1. The state variables can only change in a manner that dos not violate Equation 1.

Stationary Manifold

The second prediction of Core Model addresses populations in stationary equilibrium, the state in which the rate of natural increase, $r = b - d$ is zero and the population size is unchanged. Those countries in which the childhood death rate is not necessarily zero can be in a stationary equilibrium state that satisfies

$$b = d$$

for b and d constant in time. Core Model then predicts that the state of the system lies along the *stationary manifold*

$$v = a \tag{5}$$

$$\frac{b}{fd} = \frac{T_e}{2T^\dagger} \tag{6}$$

for countries in a stationary state. This is a restatement of Equation 49 from the Derivation section. The EOS and stationary manifolds are plotted in Figure 2. Stationary states must lie on the stationary manifold. Dynamical states in which the childhood death rate is zero lie on the EOS manifold. There is only one point E that lies on both the stationary manifold and the EOS manifold. This is the state that is stationary and has a childhood death rate of zero. This is the equilibrium point, or *stationary fixed point*, for a population with low childhood death rates. At the stationary fixed point

$$\begin{aligned} b &= d \\ f &= 1 \\ T_e &= 2T^\dagger. \end{aligned} \tag{7}$$

The Stationary Manifold and the EOS are plotted in Figure 2 and in Figure 3 along with data from OECD countries,OEC [2014a] which tend to be developed, and all other countries.CIA [2014a,b,c,d,e], OEC [2014b] When childhood deaths are sufficiently rare, the dynamical states of the model are constrained to lie on the EOS manifold of Equation 1. The OECD countries, which tend to have low childhood death rates, tend to follow the EOS manifold. Because of the similarity of this plot with similar types of plots in physics, it is referred to here to as a *phase-space plot*.

Model Dynamics

Core Model advances the state in time by two mechanisms. The first is simply a translation of Observation 3 into mathematical language. It relates the change in fertility to the median age of the population. This is the fertility assumption discussed in the Equation of State section. The second mechanism simply relates the birth rate to the number of people of reproductive age and the number of live children each woman bears.

Fertility is the demographic variable that has received the most attention in the literature. (See for example Bongaarts and Casterline [2013], Caldwell [1976], Lee [1974], Sobotka et al. [2011] .) In this paper the fertility discussion is simplified to the level of Observation 3 along with a simple mathematical translation of that observation. The assumption in the model, but not a result, is that the dynamics of Observation 3 are represented by a power law.

$$\frac{f(n)}{f(n-1)} = \left[\frac{2T^\dagger(n-1)}{T}\right]^\alpha \tag{8}$$

where n is an index for the generation, T is the maximum age a person lives, and α is an adjustable parameter referred to here as the *fertility exponent*. A childhood death rate of zero yields $T = T_e$. Equation 8 states that the fertility decreases from generation $n-1$ to generation n if the population is young, and increases if the population is old, consistent with Observation 3. Here, Equation 8 is called the *fertility dynamics*. In Core Model the maximum possible life span T is assumed constant.

It is important to note that the EOS and the stationary manifold are independent of the fertility dynamics. This can be seen trivially by observing that α does not appear in either the EOS Equation 1 or the stationary manifold Equation 5. If the EOS manifold and the stationary manifolds depended on the fertility dynamics, α would appear in those equations. The role of the fertility dynamics is to drive the state of populations to the Stationary Manifold and, for populations with zero childhood deaths, to the EOS Manifold. The fertility dynamics also drive the populations along the manifolds.

Finally, the birth rate depends on the fertility through the relationship

$$B(n) = f(n)\,[B(n-1) - Y(n-1)] \tag{9}$$

which simply states that the number of people

$$B(n) = b(n)N(n)$$

born per year into generation n is the number of people born per year $B(n-1)$ into generation $n-1$ minus the the number of young people per year

$$Y(n-1) = y(n)N(n)$$

who die in generation $n-1$ before procreating, all times the total fertility. Equation 9 specifies the number of people born in Observation 1. Here, Equation 9 is called the *birth-rate dynamics*.

The number of people who die in childhood is determined by the externally specified young death rate Y. Those people who do not die young are specified to die at some maximum age T. This is our approximation to Observation 2.

The three equations 1, 8, and 9 along with initial conditions and specification of the fertility exponent α determine the dynamical evolution of the model. The derivation of the dynamics is presented in the Derivation section and is displayed in Equations 32 through 44. The actual model dynamics derived in the Derivation section is illustrated in Figure 4. In the figure, the simulation is started in equilibrium with a childhood death rate of 70% of births and $\alpha = 1$. Note that the falling death rate caused by the decrease in childhood mortality precipitates the transition. The population size starts to rise soon after due to the larger number of surviving children. The surviving children grow to an age in which they procreate, and the birth rate increases, which increases the population size even further. The size of families with surviving children gets very large and fertility drops. Finally, the population has momentum and overshoots the new equilibrium and then relaxes to the new equilibrium. This causes the population to age temporarily before increasing fertility drives the population to the new equilibrium. This plot has the characteristic signature of the demographic transition.[Rowland, 2003, p. 17]

The phase-space trajectory is displayed in Figure 3. The childhood death rate goes to zero in one generation. The next generation moves to the new equilibrium point E, which lies on the EOS. The dynamics overshoots the equilibrium, and, in the next generation, the population moves to point T on the EOS. At point T, the population is young and has a high birth rate. The system moves downward along the EOS over a few generations back to the equilibrium point E. The dynamics once again overshoots the equilibrium and moves to the point A. The population has a low birth rate and has aged. Finally, the solution moves back along the EOS and settles into a new equilibrium E different from P. The OECD countries tend to be more developed and have lower childhood death rates than the countries that are in the early stages of the demographic transition. They also tend to follow the EOS consistent with Equation 1. Note that the dynamics moves in discrete discontinuous jumps from generation to generation. The closer the system is to the equilibrium point the more the dynamics appears to be continuous.

The trajectories for other values of Y are displayed in Figure 5. All trajectories that start in equilibrium on the stationary manifold behave similarly. They quickly move to the EOS manifold and then work their way down the manifold to the new equilibrium point which they overshoot and relax back to the new equilibrium. The higher is the initial childhood death rate the higher the trajectory climbs on the EOS manifold. In each of the trajectories, the population size increases from its initial value, overshoots the new equilibrium value, and then relaxes to its new equilibrium. This is illustrated in Figure 4.

If the childhood death rate only drops a small amount, the dynamics is constrained to the stationary manifold. As the magnitude of the jump in chilkdhood deaths increases, then the dynamics leave the stationary manifold. The case in which the jump childhood death rate is large is displayed in Figure 6. In this situation the flow is counterclockwise around the EOS manifold in the phase-space plot. Data from Gapminder[gap, 2014] developed by Rosling[Rosling, 2012] was used to plot the actual trajectories from 1953 to 2012 for a representative set of countries. This is displayed in Figure 7. The flow is counterclockwise consistent with the prediction of Core Model in Figure 6. The sequence of events start with a decrease in the death rate. This increases the ratio of births to deaths. The increase in births leads to the population becoming younger. The ferility responds to the increase of children by decreasing. This cause the population to age. This all happens on a relatively short timescale of a few generations. The state overshoots the equilibrium leading to an aged population.

Population growth in Core Model is driven by two mechanisms, (1) the responsiveness of the total fertility to pressures on the family and (2) the size of the sudden drop in childhood mortality. The first mechanism is determined by the fertility exponent α. The second is determined by the childhood death rate y. The time series for various values of α and y are displayed in Figures 9 and 8.

The exponent α in the fertility dynamics determines the responsiveness of the population to the economic, soicial, cultural, biological, and political pressures of large and small families and the ages of the family members. Large values of α represent a very responsive population. This is a population that can adjust its fertility quickly to economic shifts. This could be due to the availability of birth control, the education of women, or possibly flexibility in the timing of marriage. On the other hand, small α implies sluggish response of fertility to external economic conditions. This could be due to a lack of economic options for families. In Figures 3 and 4, α is taken to be one. In this case the dynamics overshoot the final approach to the new equilibrium. (Point A in Figure 3) This leads to a population in which the median age is greater than half the expected age—an aging population. The overshoot is simply due to the higher level of responsiveness of the population fertility to changing family economics.

Larger values of the fertility exponent α, and thus a population responsive to external pressures on the family, lead to smaller population jumps. Less responsive populations, lower values of α, lead to greater population jumps, but with smaller oscillation levels. Larger drops in childhood death rates lead to larger jumps in population size with increased levels of oscillation. Here, the ratio of final to initial population size predicted by Core Model is called *Core population momentum*. For very large values of α, greater than about 2, the dynamics becomes chaotic.

The general behavior of the dynamics of Core Model can be condensed into a few sentences.

1. The stationary manifold is an *attractor* for the system. In other words, a state of the system that does not lie on the stationary manifold moves toward the manifold. The stationary manifold is the collection of all equilibrium states for the system.

2. If the population has no childhood deaths, the state lies on the EOS manifold. The EOS manifold does not contain equilibrium states except where it intersects with the stationary manifold at point E. The states are dynamic on the EOS manifold. The system relaxes to the the intersection of the EOS and stationary manifolds, point E.

3. If the population has non-zero number of childhood deaths the state circles in a counterclockwise direction in the space of a and v.

4. The dynamics may overshoot equilibrium points, and the dynamics may move from one manifold to the other, but the dynamics tend to lie on or near the manifolds.

5. Statements 1–3 are true for moderate values of the fertility exponent and initial childhood death rate. For large values greater than approximately 2, the dynamics becomes chaotic.

The dynamics is simple enough so that it can be solved with an Excel spreadsheet. A spreadsheet is available upon request.

Classification of Populations by Their Stage in the Demographic Transition

World populations can be clustered based on their stage in the demographic transition. This is illustrated in Figure 10. There are three clusters that late in the transition. These are the Aging countries that would seem to have overshot the relaxation to equilibrium point E in Figure 5, the Uniform countries that lie near the stationary equilibrium point E, and the Young countries that appear to be approaching the stationary point E. These three clusters are characterized by their median age with respect to the expected lifetime. The populations in the Aging cluster have high median ages. The populations in the Young cluster have low median ages. The populations in the Uniform cluster have median ages that are half the expected lifetime.

There are two clusters that are earlier in the transition. These are characterized by their geography or economy. African populations tend to have low ratios of b/fd. Oil countries tend to have high ratios.

In the data set, 22 of the 36 OECD countries are in the Aging cluster. All the Aging OECD countries have such low fertility that the populations are not replacing themselves. According to Core Model, this means that most of the OECD countries are in an advanced state of the transition. Core Model predicts that the fertility in these countries will increase driving the population to an even age distribution. This has been noted in demographic observations.[Goldstein et al., 2009] These countries are enumerated in Table 1 where they are ranked according to $T_e/2T^\dagger$. Older populations are at the top of the table. Table 2 lists the countries in the Aging cluster that are not in the OECD.

There are three of 36 OECD populations that are within 1% of being completely uniform: U.K., France, and S. Korea. The data for these countries is displayed in Table 3.

There are 11 OECD countries that have young populations. These are displayed in Table 4. The most dramatic of these countries is Mexico with a high birth rate, a low death rate, high fertility, and a high ratio $b/fd = 3.35$.

There are no OECD countries in the Africa or Oil clusters. In the case of the Africa cluster, the median age is low and the ratio b/fd is also low. The Africa cluster is composed entirely of Sub-Saharan African countries. Core Model indicates that these early-transition African countries have not yet experienced significant fertility decline. This is consistent with the observations (See [Bongaarts and Casterline, 2013] for a thorough discussion).

The Oil cluster is composed mostly of countries involved with oil and natural-gas production: United Arab Emirates, Qatar, Libya, Kuwait, Brunei, Bahrain, and Singapore. The remaining countries in the cluster tend to be isolated tourism-based islands.

All other countries not mentioned tend to lie in the Young cluster. Two important populations in this cluster are China and India. China is located very near Korea, an OECD country with a uniform population. The dynamics of the Chinese population may be similar to Korea's. The closest OECD population to India is Brazil. According to Core Model, the countries in the young cluster are in advanced stages of the demographic transition and are moving towards the stationary fixed point.

Selected countries are identified in Figure 11. It should be noted that there is a tendency for populations to segregate by region. There is a Western European region that is very far along the demographic transition repesented by the EOS manifold followed in short order by the offshoot countries of USA, Australia, and New Zealand. Next come the big Asian countries of China and India, followed by Latin American countries. The Middle Eastern oil countries are off the EOS manifold as are the Sub-Saharan African countries. Islands with tourist-based economies tend to lie between the oil countries and the EOS manifold. An Excel spreadsheet of this data is available upon request.

Population Aging is a Natural Consequence of the Dynamics

Aging Non-OECD countries, with the exception of South Africa and Saint Maarten, also have low fertility. The populations are not replacing themselves. This would indicate that the Aging Non-OECD countries, other than the two exceptions, are also in late stages of transition. Of the two exceptions, South Africa has a very low value of the ratio ($b/fd = 0.98$), while Saint Maartens has a very high value ($b/fd = 2.76$). South Africa's ratio is characteristic of Serbia and Ukraine ($b/fd = 0.94$).

The Aging and Uniform populations are displayed in Figure 12 and in Tables 1, 2, and 3.

These results can be re-stated in a few intuitive sentences. If the response of fertility to external pressures is sluggish, then birth rates drop slowly in response to falling childhood death rates, leading to large population gains. If the fertility is very responsive, then the population rapidly builds momentum and overshoots equilibria, leading to population aging. In essence, population aging is the counterpoint to population growth. The balance between the two is determined by the fertility responsiveness.

Core Model does not include the effects of migration at this stage of the development of the model. Migration, however, may play a key role in rebalancing populations in aging countries.[Billari and Dalla-Zuanna, 2013] Low or very low fertility may persist across many generations without leading to population implosions.

Increased productivity may also affect rebalancing. Productivity is sustained by increasing education. Increased productivity reduces the pressure on maintaining an aged population,[Striessnig and Lutz, 2013] which may also reduce the tolerance for migration into the population. These effects are not included in Core Model.

Derivation of Core Model

Equilibrium

There are two equilibrium conditions for the system: (1)the birth rate is equal to the death rate and (2)the age distribution of the population is steady. The first condition assures that the size of the population is not growing. It is constant in time. The second condition assures that the age distribution of the population is constant in time. This paper examines how demographic systems approach equilibrium.

The steady-state equilibrium solution occurs when the rate of natural increase $r = b - d$ is zero. This defines the steady-state birth rate in terms of the death rate at steady state.

$$b_0 = d_0 \tag{10}$$

The lifetime that corresponds to a steady-state solution is designated T_0.

The median age determines a coarse age distribution for the population. Here, T is defined as the maximum age that a person may live. If the median age $T^\dagger$ is greater(less) than half the maximum age T, there are more older(younger) people than younger(older) people. The distribution in which median age is equal to half the maximum is defined, in this paper, to be a *uniform distribution.* For notational convenience, the quantity T^* is defined as twice the median age.

$$T^* = 2T^\dagger. \tag{11}$$

A condition for a uniform solution is then

$$T = T^*. \tag{12}$$

In the case in which the uniform and steady-state solutions are the same the equilibrium condition is

$$T_0 = T^* = 2T^\dagger \tag{13}$$

$$d = b = d_0 = b_0 = d^* = b^* \tag{14}$$

$$r = b - d = b_0 - d_0 = b^* - d^* = 0. \tag{15}$$

The *stable* condition, that the quantities are unchanging in time, is imposed. These, then, are the conditions for *stationary equilibrium.*

Mathematical Description of Core Model

Core Model is illustrated in Figure 14. It is assumed that there is a maximum lifespan given by T. This is the expected lifetime of someone who has survived an early death. Time is divided into discrete units of $T/2$. Actual time t is given by

$$t = \frac{nT}{2} \tag{16}$$

where n is an integer. Variation on timescales smaller than a time step are not considered.

The number of births per year at time step n is $B(n)$. The number of early deaths is $Y(n)$. Early deaths are assumed to occur immediately after birth. Birth and early deaths per year remains constant for one time step, at which time the surviving people who were born a the beginning of the time step give birth themselves. The birth rate at time step $n+1$ is then

$$B(n+1) = f(n+1)B'(n), \tag{17}$$

where

$$B'(n) \doteq B(n) - Y(n), \tag{18}$$

and $f(n)$ is the total fertility at time n. Fertility is defined such that $f = 1$ is the replacement rate, the amount of children born per person (male or female) to just replace the population. Equal numbers of males and females are assumed.

The number of people who were born in time step n and survived early death live until their maximum age T, at which time they die. The total number of deaths per year D at time step $n+2$ is then

$$D(n+2) = B'(n) \tag{19}$$

and the total population N at time step $n+2$ is

$$N(n+2) = B'(n+1)\frac{T}{2} + B'(n)\frac{T}{2}. \tag{20}$$

The birth b and death d rates per person per year at time n are

$$b(n) = \frac{B(n)}{B'(n-1) + B'(n-2)}\frac{2}{T} \tag{21}$$

and

$$d(n) = \frac{B'(n-2)}{B'(n-1) + B'(n-2)}\frac{2}{T}. \tag{22}$$

The ratio of birth to death rates at time n is

$$\frac{b(n)}{d(n)} = \frac{B(n)}{B'(n-2)} = \frac{f(n)B'(n-1)}{B'(n-2)}. \tag{23}$$

The median age of a population is the age at which half the population is younger and half is older. If the number of younger people exceeds the number of older people, then the median age in our model is less than $T/2$. The condition for $T^\dagger$ to be the median age is then

$$B'(n-1)T^\dagger(n) = B'(n-1)\left(\frac{T}{2} - T^\dagger(n)\right) + B'(n-2)\frac{T}{2} \tag{24}$$

which yields

$$\frac{B'(n-1)}{B'(n-2)} = \frac{T}{4T^\dagger(n) - T} \qquad T^* = 2T^\dagger < T. \tag{25}$$

In the limit of zero childhood deaths, Equation 25 becomes

$$\frac{b(n)}{f(n)d(n)} = \frac{T_e}{4T^\dagger(n) - T_e} \quad T^* = 2T^\dagger < T \quad y \to 0. \tag{26}$$

Similarly, for $T^\dagger \geq T/2$ yields

$$B'(n-1)\frac{T}{2} + B'(n-2)\left(T^\dagger(n) - \frac{T}{2}\right) = B'(n-2)\left(T - T^\dagger\right) \tag{27}$$

which yields

$$\frac{B'(n-1)}{B'(n-2)} = \frac{3T - 4T^\dagger(n)}{T} \quad T^* = 2T^\dagger \geq T. \tag{28}$$

In the limit of zero childhood deaths, Equation 28

$$\frac{b(n)}{f(n)d(n)} = \frac{3T_e - 4T^\dagger(n)}{T_e} \quad T^* = 2T^\dagger \geq T \quad y \to 0. \tag{29}$$

The life expectancy at birth, T_e, for people born at time n for our model is proportional to the number of people born at time n who survive to age T over the total number of people who were born into that cohort including those that died shortly after birth.

$$B(n)T_e(n) = [B(n) - Y(n)]\,T$$

$$\frac{T_e(n)}{T} = \frac{B'(n)}{B(n)} = \frac{B(n) - Y(n)}{B(n)}. \tag{30}$$

In the case that the early death rate Y is zero, the life expectancy, T_e, is equal to the maximum lifespan, T, Equations 25 and 28 reduce to Equation 1.

If it is assumed that the driver of the dynamics is the early death rate Y, then the time evolution of the demographic variables is completely specified by the above equations with the exception of the fertility f. The fertility is not yet specified. Specification of the fertility closes the mathematical model. The measure of the number of children per family, within the context of Core Model, is the median age. If the median age is low compared with lifespan, then there are many children per family. Conversely, if the median age is high, there are few children on average per family. The following relationship between fertility and median age is imposed.

$$f(n+1) = \left(\frac{T^*(n)}{T}\right)^\alpha f(n) \tag{31}$$

Here, α is an adjustable parameter. For positive α, this relationship lowers the fertility when the median age is low and raises it for high median age. This relationship closes the model and makes it self-consistent. All the details of the economics that drive fertility rates are encapsulated in the exponent α. If α is large, then the population has the ability to respond quickly to economic forces. The sudden availability of cheap birth control, for instance, can increase the value of α thus increasing the ability of the population to respond to the stresses of large family size. Conversely, if α is small, the population does not have the ability to respond to the stresses of large or small families.

Given a set of initial conditions and the early death rate as an external driver, Core Model can be evolved in time. The initial conditions are given when two consecutive values of the birth rate $B(0)$ and $B(1)$, the median age $T^*(1) = 2T^\dagger(1)$, and the fertility $f(1)$ are specified. The external driver of the model is the death rate for early death Y. The driver is specified when its values are given for all time. The evolution is then

$$\frac{T^*(n)}{T} = F(B') \tag{32}$$

where

$$\begin{aligned} F(B') &= \frac{1}{2}\left[1 + \frac{B'(n-2)}{B'(n-1)}\right] \quad T^* < T \\ &= \frac{1}{2}\left[3 - \frac{B'(n-1)}{B'(n-2)}\right] \quad T^* \geq T. \end{aligned} \tag{33}$$

and the remaining finite difference equations are

$$f(n) = \left(\frac{T^*(n-1)}{T}\right)^{\alpha} f(n-1) \tag{34}$$

$$B(n) = f(n)B'(n-1) \tag{35}$$

with the auxiliary relationships

$$T^{\dagger} = \frac{T^*}{2} \tag{36}$$

$$B'(n) = B(n) - Y(n) \tag{37}$$

$$y(n) \doteq \frac{Y(n)}{N(n)} \tag{38}$$

$$D(n) = B'(n-2) \tag{39}$$

$$N(n) = \left(\frac{B'(n-1) + B'(n-2)}{2}\right) T \tag{40}$$

$$b(n) = \frac{B(n)}{N(n)} \tag{41}$$

$$d(n) = \frac{B'(n-2)}{N(n)} \tag{42}$$

$$\frac{T_e(n)}{T} = \frac{B'(n)}{B(n)} = \frac{B(n) - Y(n)}{B(n)} = \frac{b(n) - y(n)}{b(n)} \tag{43}$$

and

$$\begin{aligned} F(B') &= \frac{1}{2}\left[1 + \frac{f(n)d(n)}{b(n)}\right] \quad T^* < T,\ Y = 0 \\ &= \frac{1}{2}\left[3 - \frac{b(n)}{f(n)d(n)}\right] \quad T^* \geq T,\ Y = 0. \end{aligned} \tag{44}$$

Equations 32, 34, and 35 define the time evolution of the system. They comprise a set of nonlinear finite-difference equations. The equation of state Equation 1 follows from Equations 32 and 44.

Core Model Description of the Demographic Transition

The dynamics are determined by the initial conditions and temporal specification of the early death rate. The system starts out in equilibrium at a high rate of early death Y. The early death rate is assumed to drop to zero within a single generation. This can be due to external forces such as the discovery of antibiotics, creation of clean water supplies, or implementation of vaccinations. The early death rate then remains zero while the system evolves to a new equilibrium.

The pre-transition stationary-equilibrium state is given by

$$b = d \tag{45}$$

$$T = T^* \tag{46}$$

$$b = f(b - y) \tag{47}$$

$$\frac{T_e}{T} = \frac{b - y}{b} \tag{48}$$

which implies

$$\frac{b}{fd} = \frac{T_e}{T^*}. \tag{49}$$

This is the stationary manifold of Equation 5 in the results section. All stationary states lie on this manifold.

Without loss of generality the inital values for $B(0) = B(1)$ are taken to be one.

$$B(1) = B(0) = 1 \tag{50}$$

The early death rate Y is externally given and is some fraction of B. The initial value of the total fertility is then

$$f(1) = \frac{B(1)}{B(0) - Y(0)} \tag{51}$$

which follows from equation 35. The initial age distribution is uniform. This implies

$$\frac{T^*(1)}{T} = 1. \tag{52}$$

This is the complete set of initial conditions. As long as Y remains constant, all quantities in the model are constant at their initial values.

At some time after the initial period, the early death rate Y is dropped to zero. The drop in childhood death rate is assumed to take place within one generation. The case for $\alpha = 1$ and childhood death rate Y equal to 70% of births is displayed in Figures 3 and 4. The state moves onto the EOS manifold within a generation. The time series has the characteristics of the demographic transition. The transition is lead by falling mortality, followed by an increasing, then decreasing birth rate. Finally the fertility drops. During the transition the population increases because the birth rate exceeds the death rate. The dynamics predict, for $\alpha = 1$, that the trajectory overshoots the equilibrium, which leads to an aging population in late-transition countries.

Discussion

The demographic transition, as characterized, is an extreme event.[Casti, 2012] The population cycle begins life in a slowly changing state. Pressures on the system that are external to the system increase. The system is perturbed and undergoes a dramatic change. Finally the systems relaxes to a new slowly changing state where the process is repeated. The life cycle of an extreme event is displayed in Figure 15.

In the case of the demographic transition, a population starts in a near steady state in which the birth rate is approximately equal to the death rate. The death rate for people who have yet to procreate, the childhood death rate, is significant. The fertility, at this stage, is greater than the death rate in order to replace the children who died. The common observation is that a sudden drop in the childhood death rates triggers the extreme event, the demographic transition.[Dyson, 2010] The drop in childhood deaths starts a cascade of events that culminates in the system returning to a steady state different from the original steady state. This is a rebalancing of the population.

Core Model is a simple accounting of births and deaths with an added assumption on how the fertility change is related to age distribution in the population. The key speculative causal relationship in the model is associated with fertility, perhaps the most difficult concept in demographics. The change in fertility is assumed to depend on the age distribution of the population and on one free parameter, α, that determines the speed at which fertility responds to the age distribution. The free parameter is assumed to depend on external economic, sociological, cultural, political, and biological drivers that do not appear explicitly in the model. More precisely, the free parameter may be a function of non-demographic variables such as the level of women's education, income, urbanization, level of healthcare, family structure, or any of a number of external variables.[Rosling, 2012] The fertility assumption can be expressed mathematically as (Equation 8)

$$\frac{f(n)}{f(n-1)} = \left[\frac{2T^{\dagger}(n-1)}{T}\right]^{\alpha}$$

where f is the fertility, n is the generation number, T is the maximum possible life span, $T^{\dagger}$ is the median age, and α is the free parameter. For low values of α the movement through the demographic transition is sluggish. The fertility is slow to respond to the shifting age distribution. The population experiences significant growth because fertility is dropping slowly. This is the situation we might expect if women's education is low or the family income is low. As α becomes large, the dynamics becomes very responsive to changes in age distribution. The population size overshoots the steady state values leading to aging populations. This is a situation in which women are more in control of their reproduction. For values pf α greater than approximately two, fertility is very responsive to changes in age distribution. In this case the dynamics becomes chaotic.

The purpose of any model is to make testable predictions. If the model is based on a set of causal relationships, a successful test of the model reinforces the confidence in the causal relations.

One prediction of Core Model is that the temporal ordering of events; reduced death rate, increased birth rate, younger population, population growth, reduced fertility, reduced birth rate, aging population, and, finally, saturation of population growth at a greater population size; is consistent with observations of the temporal ordering in actual populations.

A second prediction is that population aging is a consequence of how rapidly the the childhood death rate falls and how responsive fertility is to changing age distributions in the population. The aging population occurs a few generations after the childhood death death drops dramatically. The actual aging of populations late in the transition, most notably in European populations, is consistent with this prediction. If the model is correctly predictive, then the overshoot is due to the degree and suddenness of the drop in the childhood death rate and the responsiveness of the fertility to the changing age distribution. After aging has occurred, the model predicts that fertility will increase, and the age distribution will become uniform. If the change in childhood death rate is small, then the overshoot and consequently the level of aging in post-transition is also small. Core Model, however, does not yet include the effects of migration, which may be a major player in population rebalancing in aging and developed countries. The model also does not include the effects of increased education, which may lead to increased productivity, which may release economic pressures on aging populations. This released pressure may affect the tolerance for migration into the population.

A third prediction is that all countries on the planet have entered the demographic transition. OECD countries are in the later stages of the transition, while Sub-Saharan African countries are in early stages. The countries cluster by region based on where they are in the transition, with Europe, the offshoot countries of the U.S. and Australia, and Asian countries in the later stages of transition. Latin American and the Oil countries are in an intermediate stage. Finally, Africa is in an early stage. The dynamic states of the populations are attracted to curves defined by the model known as the Stationary Manifold and the Equation of State. There are two important dimensionless quantities that define the stage of transition, one associated with the level of growth of the population size, the other associated with the age distribution in the population.

A fourth prediction is that the gross features of the demographic transition can be described by a closed model that includes only demographic variables. External economic, sociological, cultural, political, and biological drivers affect only the childhood death rate and the fertility.

The success of these predictions increases our confidence in the hypothesized causal relationships in the model.

The model is subject to criticism. Core Model does not accurately treat timescales shorter than a lifetime. All of sociology and economics is buried in external drivers and a free parameter. Migration is not included in the model, nor is education and productivity. Core Model assumes the maximum possible life span is a constant. This may not be accurate. The ratio of life expectancy to median age is a coarse measure of population aging. The approach taken in this paper is not the standard research approach in demographics. These are open discussion and research items. Despite the possible criticisms, Core Model attempts to capture the essence of the demographic process, and seems to provide some insight into and predictive power for the demographic transition.

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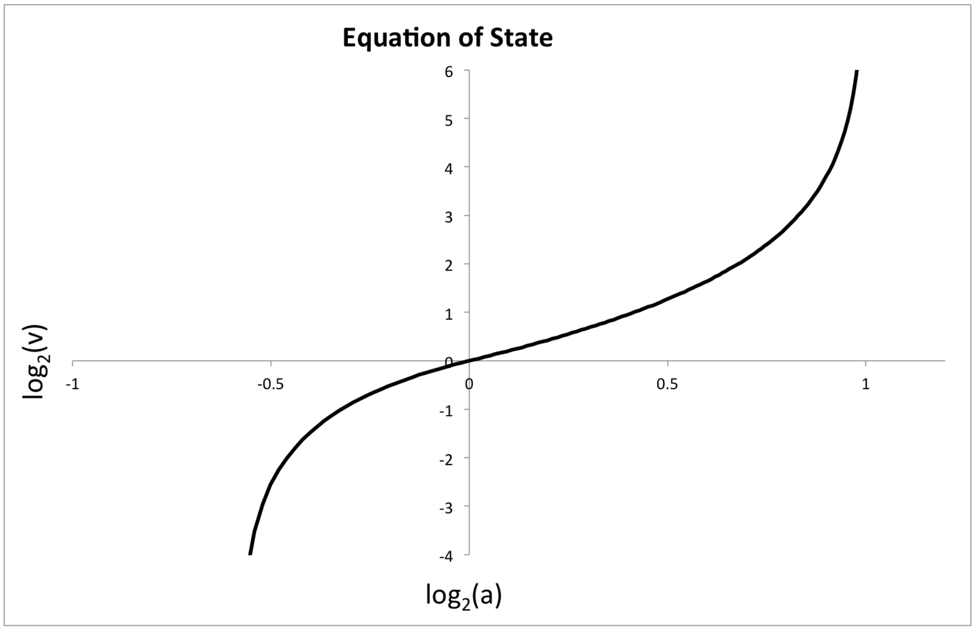

Figure 1. Equation of State. The variables are plotted as logarithms to the base 2. The origin is the point in which $v = a = 1$. The EOS log plot approaches infinity when $a \to 2$ or, equivalently, when $\log_2(a) \to 1$. The EOS log plot approaches minus infinity when $a \to 2/3$ or, equivalently, when $\log_2(a) \to -0.5849$. The EOS is undefined outside the boundaries of $a \in (2/3, 2)$.

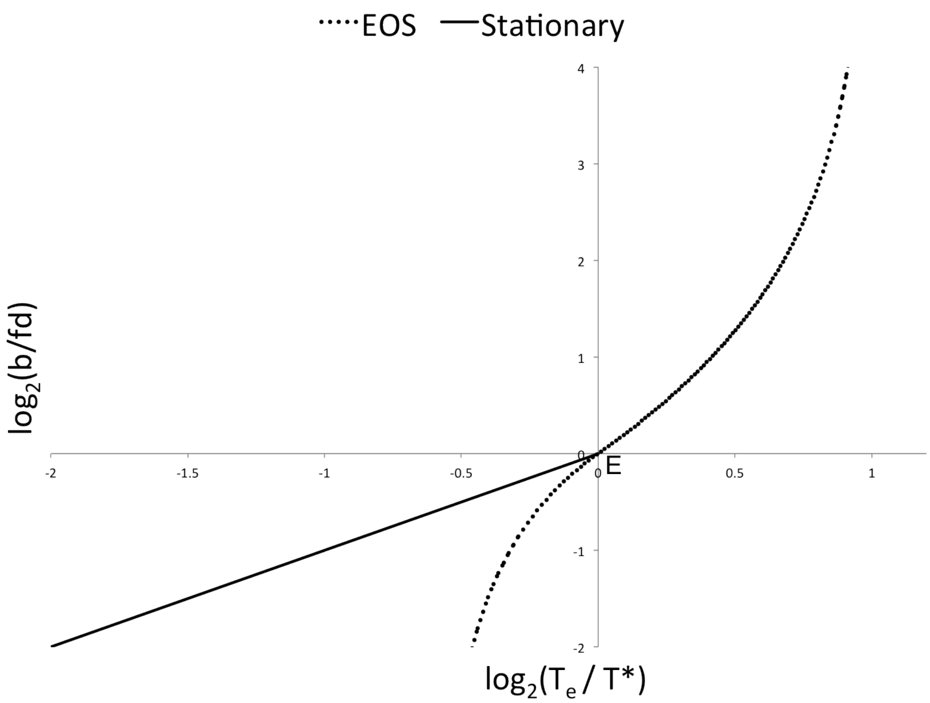

Figure 2. EOS and Stationary Manifolds. Here, $T^* \doteq 2T^\dagger$. The EOS manifold is plotted as a dotted line. The stationary manifold is plotted as a solid line. The point E is the common point on both manifolds. If the population starts out in stationary equilibrium with a non-zero childhood death rate, the initial state will lie on the stationary equilibrium manifold. If the childhood death rate drops to zero within one generation, the population will jump to the EOS manifold. The system will then move along the EOS manifold to relax at the new equilibrium point E.

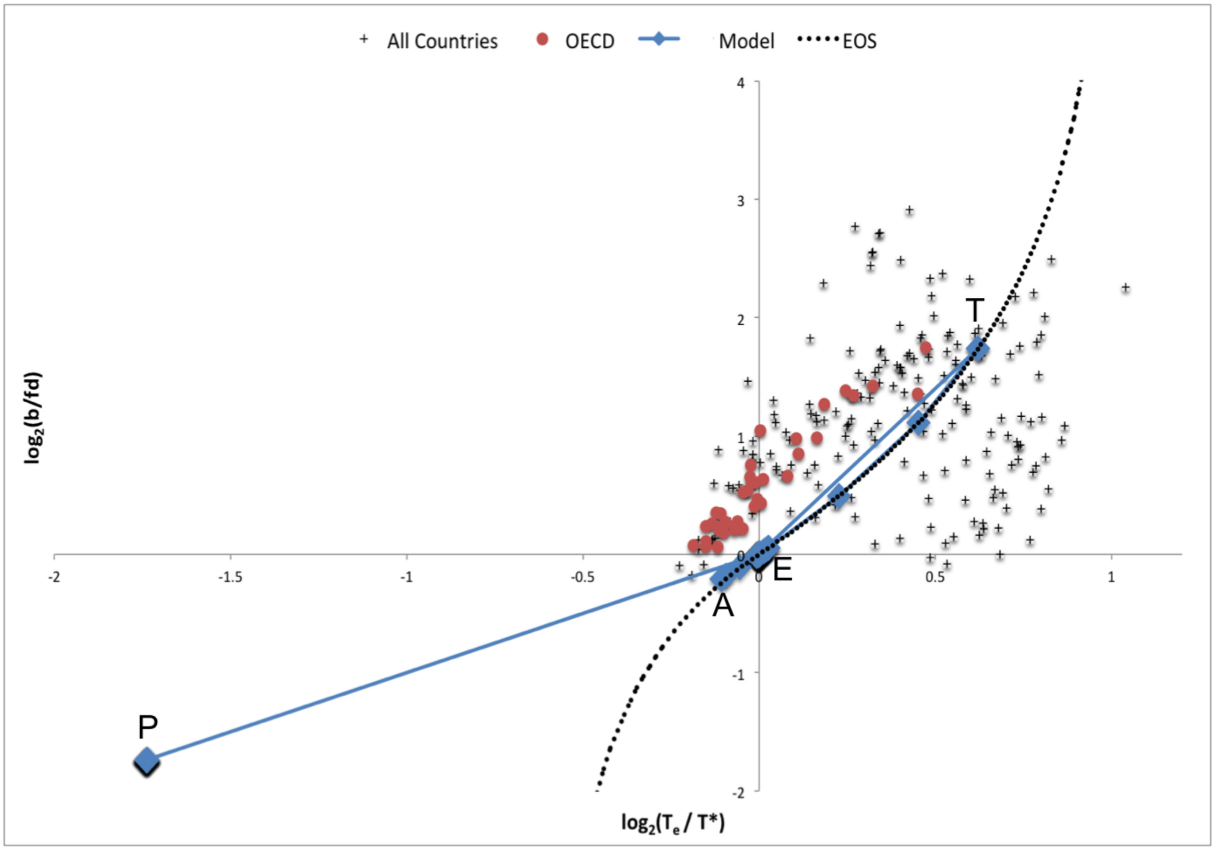

Figure 3. Phase-Space Plot. Here, $T^* \doteq 2T^\dagger$. The equation of state (EOS) is the dotted curve. The solution of the Core Model dynamics is plotted as blue diamonds. The OECD countries are plotted as red circles. The remainder of the countries on the planet are plotted as pluses. The dynamics start at point P in equilibrium on the stationary manifold with a childhood death rate of 70% of the birth rate. The exponent α is taken to be one. The OECD countries tend to be more developed and have lower childhood death rates than the countries that are in the early stages of the demographic transition. They also tend to lie near the EOS manifold. The logarithms are taken to the base 2.

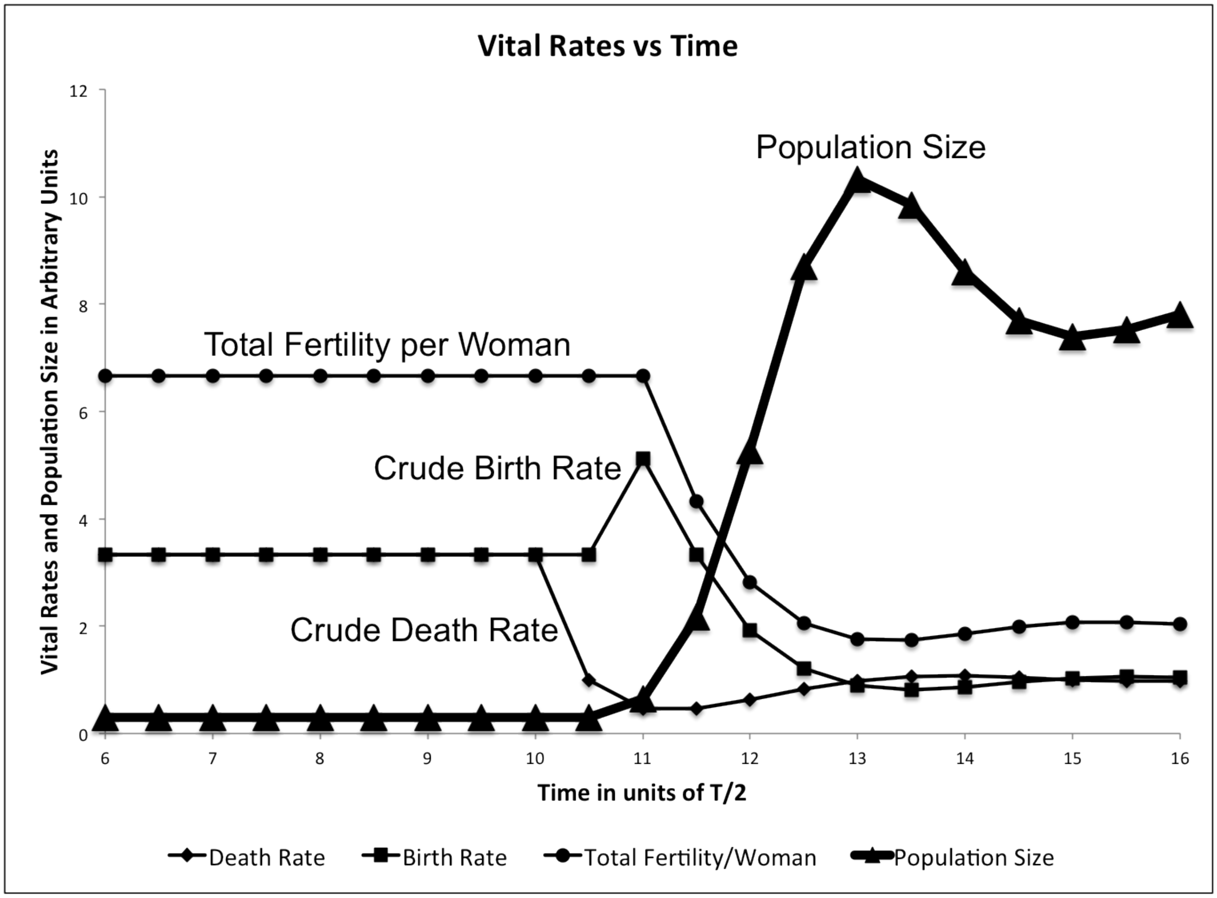

Figure 4. Time-Series of Vital Rates. Here, $\alpha = 1$, and Y is taken initially to be 70% of B. The units of time are one generation. The birth, death, and fertility are in arbitrary units, although they are in proper proportion to each other. The population size is in arbitrary units.

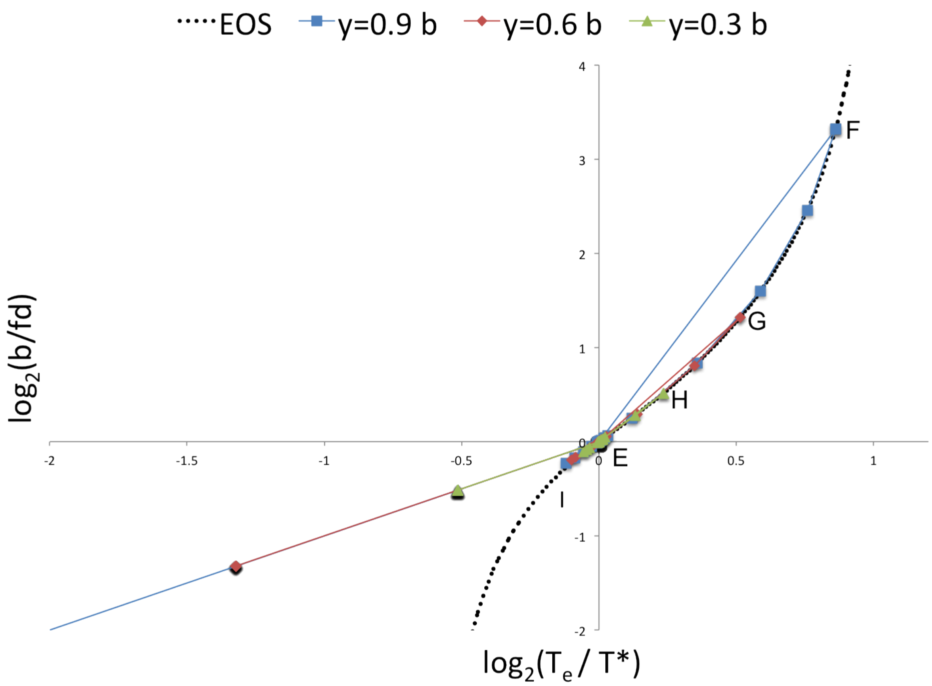

Figure 5. Trajectories for various values of childhood death. Here, $y = Y/N$ and $T^* = 2T^\dagger$. The trajectory for initial value of $y = 0.9b$ is plotted with blue square symbols; for $y = 0.6b$ is plotted with red diamond symbols; and $y = 0.3b$ is plotted with green triangle symbols. In each trajectory, the childhood death rate y drops to zero in one generation. Each trajectory starts in equilibrium on the stationary manifold. In one generation after the childhood death rate goes to zero, all trajectories go to the new equilibrium point E, and then overshoot it to lie on the EOS manifold. The overshoot increases with y (points H, G, and F). The trajectories march along the EOS manifold back to the new equilibrium E where they overshoot once more to points I on the EOS manifold. The system then finally relaxes to point E, the stationary fixed point for zero childhood deaths.

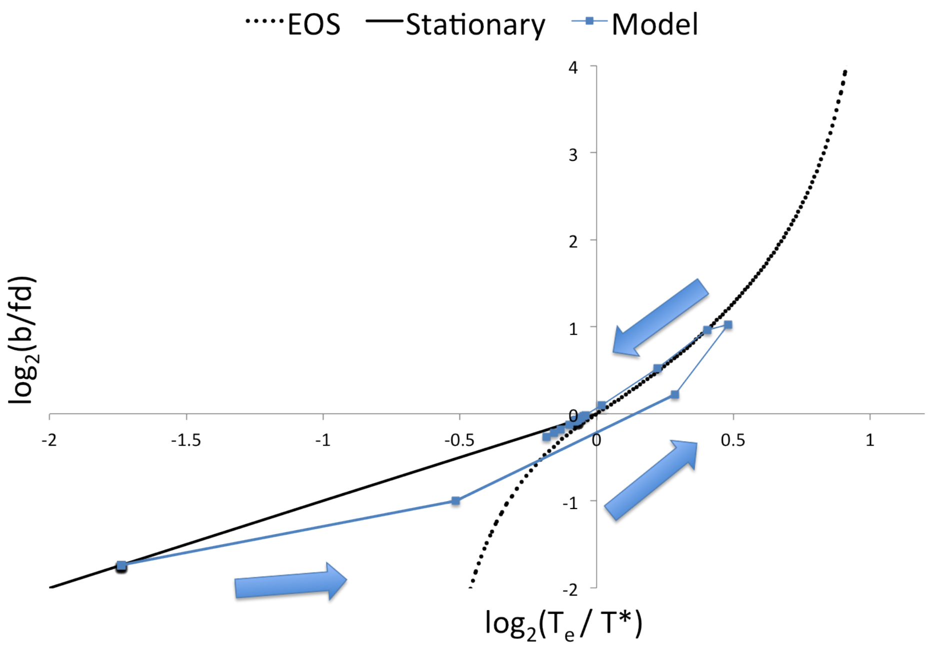

Figure 6. Phase-space plot for the situation in which the childhood death rate drops from $0.7b$ to $0.3b$. Here, α is taken to be one. Note the counterclockwise flow.

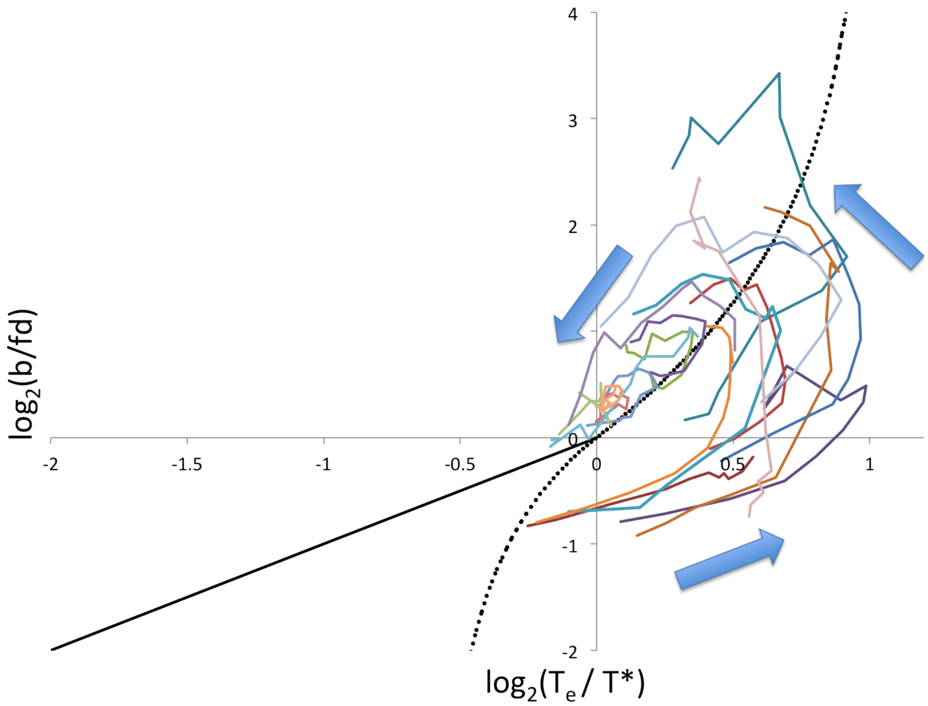

Figure 7. Phase-space trajectories for a representative sampling of countries. The time ranges from 1953 to 2013. Note the counterclockwise flow consistent with Core Model in Figure 6. The sequence of events start with a decrease in the death rate. This increases the ratio of births to deaths. The increase in births leads to the population becoming younger. The fertility responds to the increase of children by decreasing. This cause the population to age. This all happens on a relatively short timescale of a few generations. The state overshoots the equilibrium leading to an aged population. The state then relaxes to an equilibrium. The The countries represented are Chad, Kenya, Kuwait, Saudi Arabia, Mexico, Brazil, USA, Australia, China, India, France, UK, Germany, Japan, Russia, Sweden, Singapore, and Qatar.

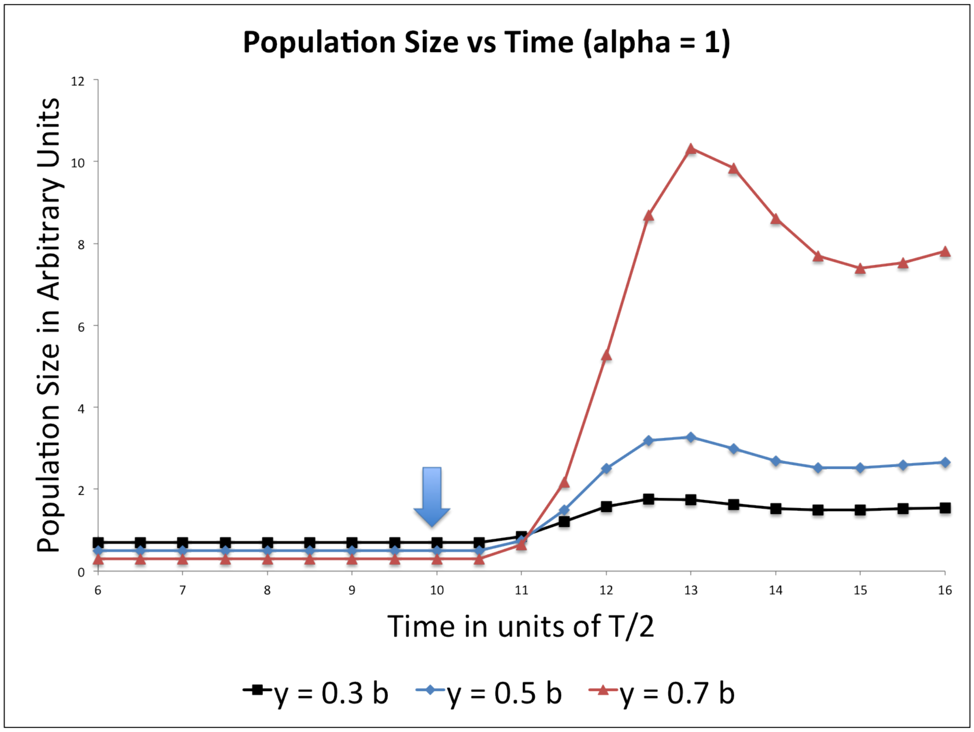

Figure 8. Population time series for $\alpha = 1$ and $y = 0.7b, 1.0b, 1.3b$. Large jumps in childhood death rate lead to large jumps in population size and oscillations. The childhood death rate drops to zero at $t = 10$. The time between markers is one generation $T/2$. All three trajectories lie on the EOS manifold (not shown).

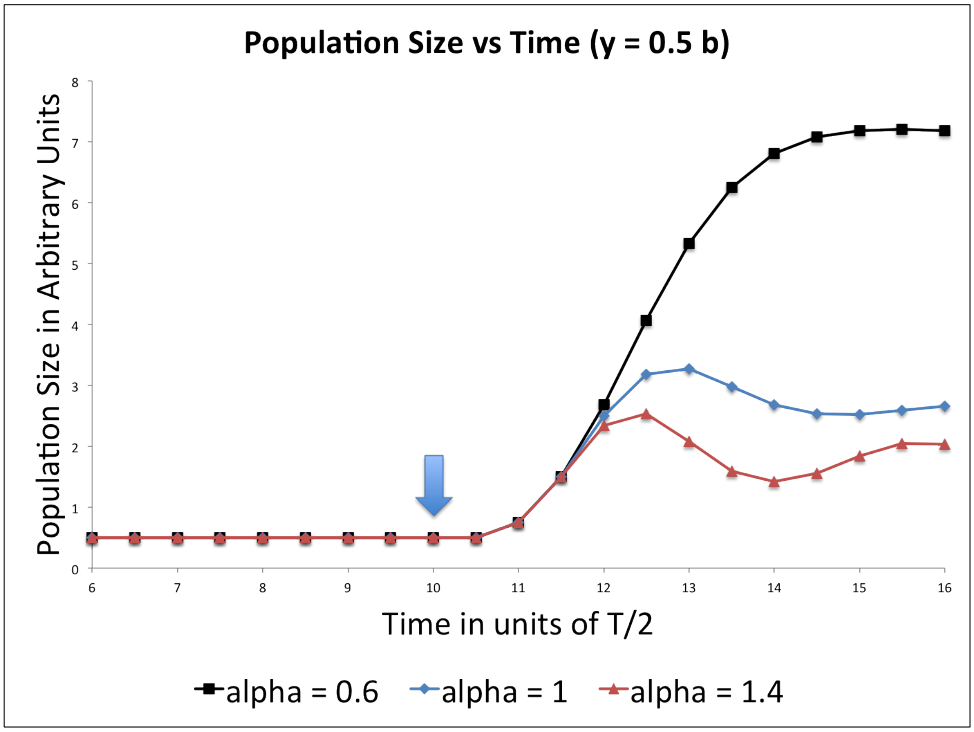

Figure 9. Population time series for $y = 0.5b$ and $\alpha = 0.6, 1.0, 1.4$. Final population size is greatest for low values of α and lowest for high values of α. Oscillation amplitudes are greater for high values of α. The childhood death rate drops to zero at $t = 10$. The time between markers is one generation $T/2$. All three trajectories lie on the EOS manifold (not shown).

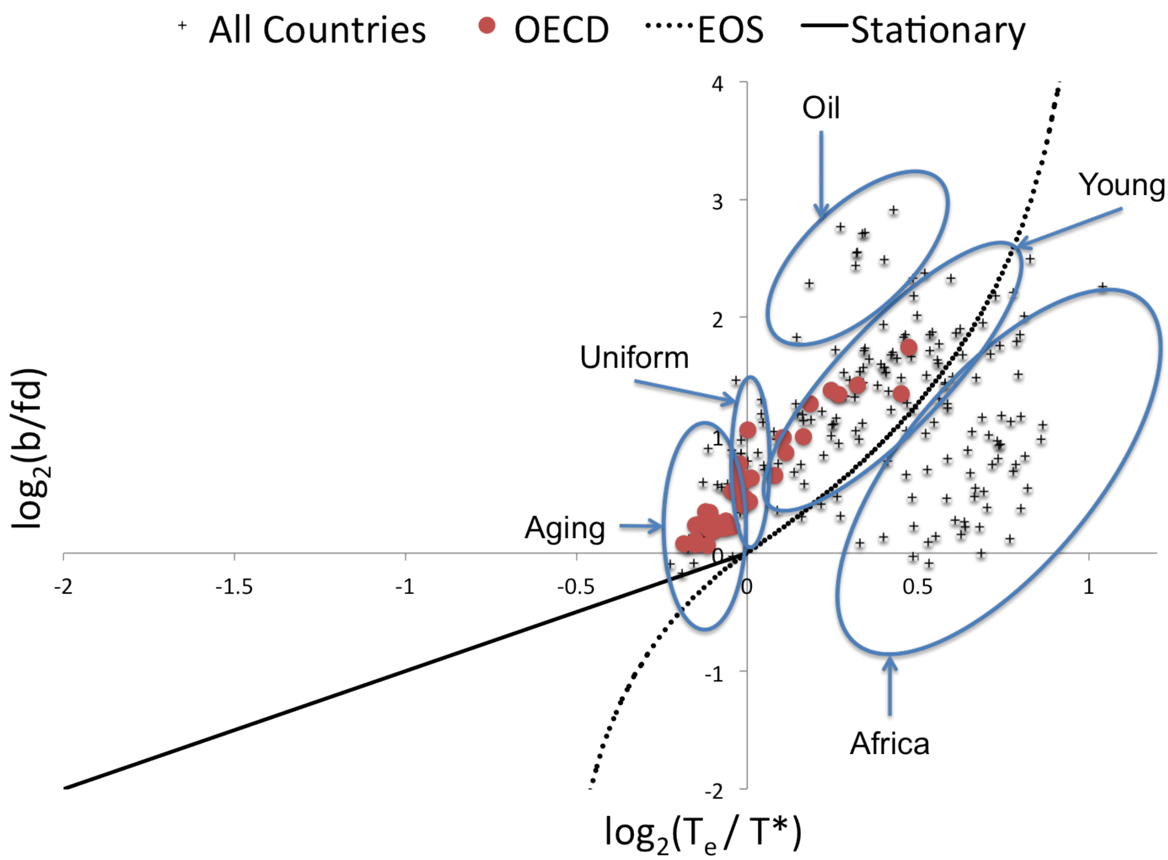

Figure 10. Clustering by Dynamics. The world's populations can be classified into five structures based on dynamics: Aging, Uniform, Young, Africa, and Oil.

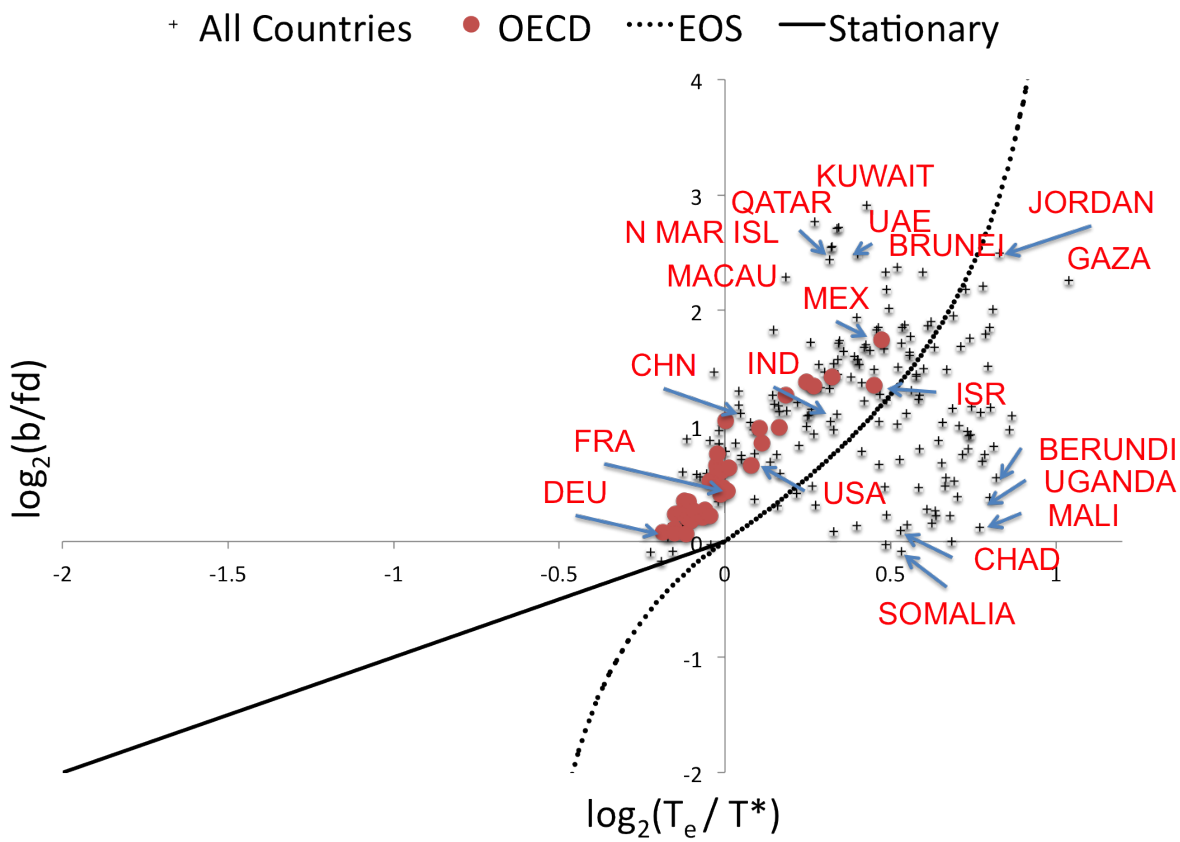

Figure 11. Selected Countries. Note that the populations tend to aggregate by geographical region.

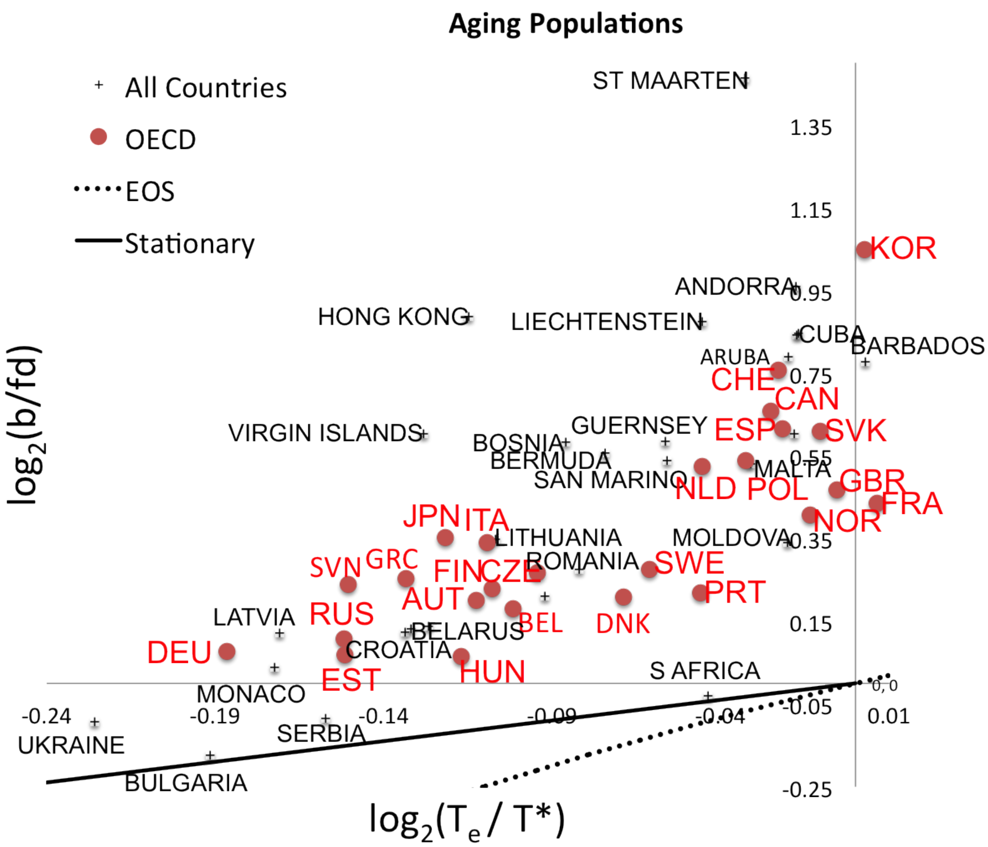

Figure 12. Blowup of Aging and Uniform Countries. European countries make up the bulk of aging populations. Hong Kong and Japan are notable exceptions. Korea is a uniform population. There are also a few island populations that are aging: Guernsey, Bermuda, Virgin Islands, Cuba, and St. Maarten. Barbados is a uniform population. There is one aging African country, South Africa.

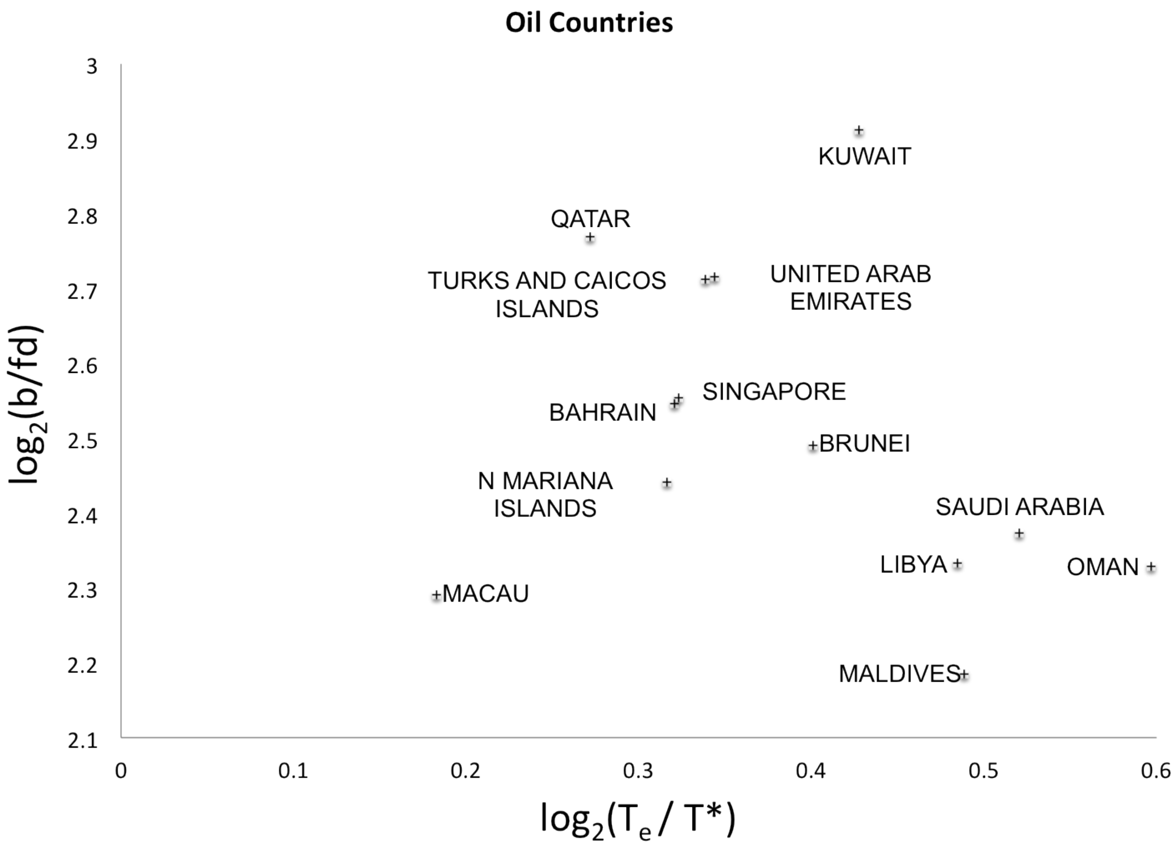

Figure 13. Blowup of Oil countries.The Oil countries lie off the EOS manifold. This is displayed in the cluster marked 'Oil' in Figure 10. There are a few island populations with tourism-based economies that share the cluster with Oil countries: Turks and Caico Islands, Northern Mariana Islands, Macau, and the Maldives. These islands lie between the Oil countries and the stationary fixed point.

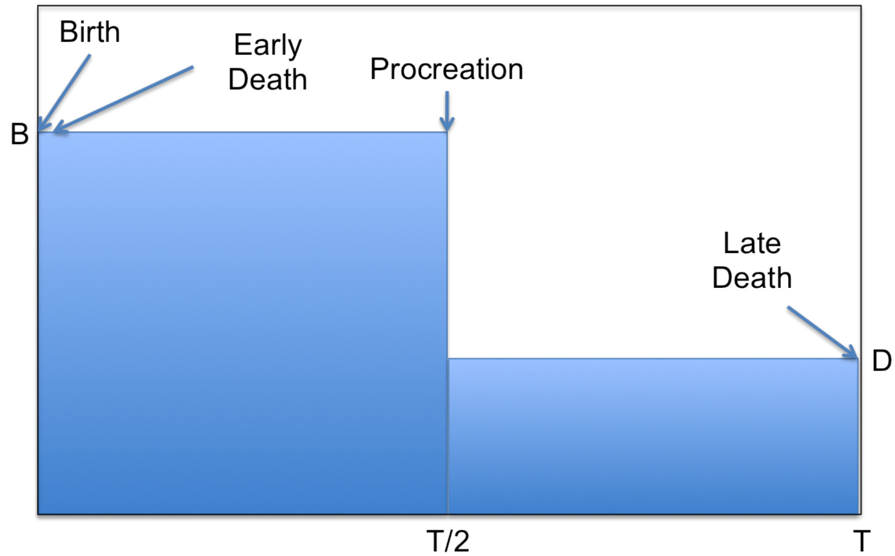

Figure 14. Core Model. People have three roles: to be born, to procreate, and to die. Some die early. In Core Model it is assumed that early deaths occur soon after birth. The remainder of the population live to their expected lifetime (after surviving early death) T at which time they die. At the halfway point in life $T/2$, people give birth to children. Here, $T/2$ is the time step, which is called, in this paper, a *generation*. Time is expressed as $t = nT/2$ where n is an integer. The number of children born per year during time step n is $B(n)$. The number of people who die early during time step n is $Y(n)$. The number who die late is $D(n) = B(n-2) - Y(n-2) \doteq B'(n-2)$. The number of children a person has at time step n is$f(n)$. The replacement rate is $f = 1$. The total population at timestep n is $N(n) = B'(n-1)T/2 + B'(n-2)T/2$.

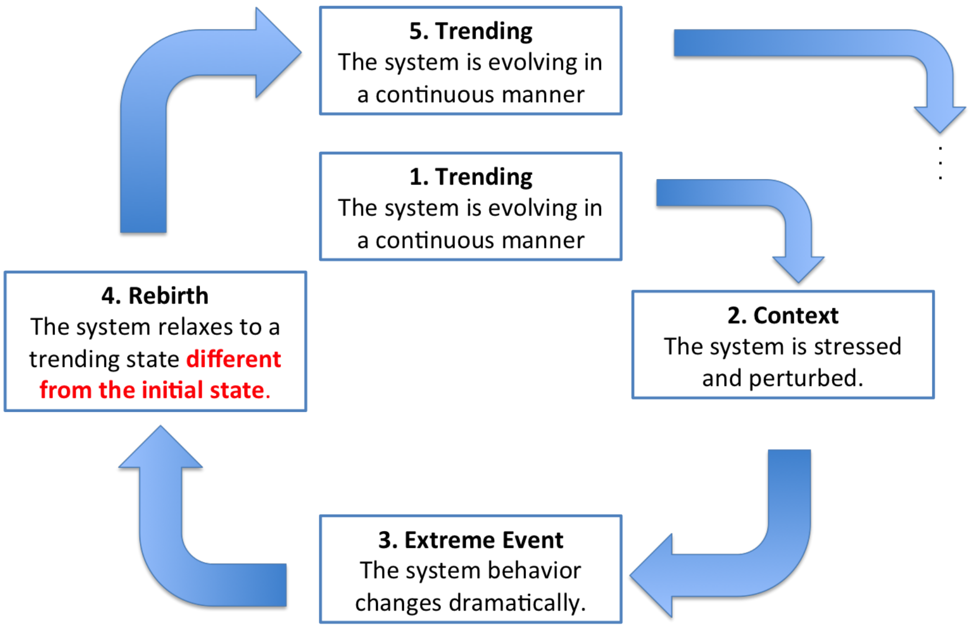

Figure 15. Life cycle of an extreme event.

List of Tables

Table 1. Aging OECD Countries

Country	ID	$b \times 1000$	$d \times 1000$	$2f$	T_e (yrs)	$T^\dagger$ (yrs)	$T_e/2T^\dagger$	b/fd
Germany	DEU	8.37	11.17	1.42	80.8	45.7	0.88	1.06
Russia	RUS	12.11	13.97	1.61	69.8	38.8	0.90	1.08
Estonia	EST	10.38	13.65	1.45	76.3	41	0.90	1.05
Slovenia	SVN	8.66	11.12	1.32	80.1	43.1	0.90	1.18
Austria	AUT	8.73	10.31	1.42	81.1	43.9	0.91	1.19
Japan	JPN	8.23	9.27	1.39	82.7	45.8	0.92	1.28
Hungary	HUN	9.37	12.71	1.41	75	40.8	0.92	1.05
Finland	FIN	10.36	10.42	1.73	80.6	43	0.93	1.15
Italy	ITA	8.94	10.01	1.41	82.7	44.2	0.93	1.27
Greece	GRC	8.94	10.9	1.40	80.7	43.2	0.93	1.17
Belgium	BEL	10	10.7	1.65	80.5	42.8	0.93	1.13
Czech	CZE	8.55	11.01	1.29	78	41.4	0.94	1.20
Denmark	DNK	10.2	10.21	1.73	79.9	41.4	0.95	1.15
Sweden	SWE	10.33	10.22	1.67	81.9	42.4	0.96	1.21
Portugal	PRT	9.59	10.91	1.51	80.8	40.7	0.97	1.16
Netherlands	NLD	10.85	8.48	1.78	81.3	41.8	0.97	1.44
Poland	POL	9.88	10.31	1.32	76.9	39.1	0.98	1.45
Canada	CAN	10.28	8.2	1.59	81	41.5	0.98	1.58
Switzerland	CHE	10.45	8.08	1.53	82.8	41.8	0.98	1.69
Spain	ESP	10.14	8.94	1.48	82.4	41.3	0.99	1.53
Norway	NOR	10.8	9.21	1.77	81.4	40.6	0.99	1.33
Slovak Republic	SVK	10.27	9.69	1.39	76.1	38.4	0.99	1.52

Table 2. Aging Non-OECD Countries

Country	$b \times 1000$	$d \times 1000$	$2f$	T_e (yrs)	$T^{\dagger}$ (yrs)	$T_e/2T^{\dagger}$	b/fd
Ukraine	9.52	15.75	1.29	68.93	40.3	0.86	0.94
Bulgaria	9.07	14.31	1.43	74.08	42.3	0.88	0.89
Monaco	6.79	8.75	1.51	89.63	50.5	0.89	1.03
Latvia	9.91	13.6	1.34	73.19	41.2	0.89	1.09
Serbia	9.15	13.77	1.41	74.79	41.7	0.90	0.94
Croatia	9.53	12.06	1.45	76.2	41.8	0.91	1.09
Saint Pierre and Miquelon	7.79	9.18	1.55	80.13	43.9	0.91	1.09
Virgin Islands	10.69	7.95	1.77	79.61	43.5	0.92	1.52
Belarus	10.99	13.68	1.46	71.81	39.2	0.92	1.10
Hong Kong	7.58	7.39	1.11	82.2	44.5	0.92	1.85
Lithuania	9.36	11.48	1.28	75.77	40.8	0.93	1.27
Isle of Man	11.27	9.99	1.95	80.87	43.1	0.94	1.16
Bosnia and Herzegovina	8.92	9.53	1.25	76.12	40.4	0.94	1.50
Romania	9.4	11.86	1.31	74.45	39.4	0.94	1.21
Bermuda	11.39	7.9	1.96	80.93	42.6	0.95	1.47
Guernsey	9.95	8.61	1.54	82.32	42.8	0.96	1.50
San Marino	8.78	8.17	1.48	83.12	43.2	0.96	1.45
Liechtenstein	10.67	6.89	1.69	81.59	42.1	0.97	1.83
South Africa	19.14	17.36	2.25	49.48	25.5	0.97	0.98
Saint Maarten	13	4.51	2.09	77.61	39.7	0.98	2.76
Georgia	10.72	10.17	1.46	77.51	39.6	0.98	1.44
Moldova	12.38	12.61	1.55	69.82	35.4	0.99	1.27
Aruba	12.72	8	1.84	76.14	38.6	0.99	1.73
Malta	10.27	8.84	1.53	79.98	40.5	0.99	1.52
Andorra	8.88	6.67	1.37	82.58	41.8	0.99	1.94
Cuba	9.92	7.58	1.46	78.05	39.5	0.99	1.79
St Helena, Ascension, & T. da Cunha	10.19	7.22	1.57	79.06	40	0.99	1.80

Table 3. Uniform OECD Countries

Country	ID	$b \times 1000$	$d \times 1000$	$2f$	T_e (yrs)	$T^{\dagger}$ (yrs)	$T_e/2T^{\dagger}$	b/fd
UK	GBR	12.26	9.33	1.90	81.1	40.3	1.00	1.38
Korea	KOR	8.33	6.5	1.24	81.1	39.7	1.00	2.07
France	FRA	12.6	8.96	2.08	82.2	40.6	1.00	1.35

Table 4. Young OECD Countries

Country	ID	$b \times 1000$	$d \times 1000$	$2f$	T_e (yrs)	$T^\dagger$ (yrs)	$T_e/2T^\dagger$	b/fd
Luxembourg	LUX	11.72	8.52	1.77	81.1	39.6	1.01	1.55
US	USA	13.66	8.39	2.06	78.7	37.2	1.06	1.58
Australia	AUS	12.23	7.01	1.77	82	38.1	1.08	1.97
New Zealand	NZL	13.48	7.25	2.06	81.2	37.4	1.08	1.81
Iceland	ISL	13.15	7.07	1.88	82.4	36.2	1.12	1.98
Ireland	IRL	15.5	6.41	2.01	80.6	35.4	1.14	2.41
Chile	CHL	14.12	7.31	1.85	78.3	33	1.19	2.60
Brazil	BRA	14.97	6.51	1.81	73.5	30.3	1.20	2.54
Turkey	TUR	17.22	6.11	2.1	74.6	29.2	1.25	2.68
Israel	ISR	18.71	5.52	2.65	81.8	29.7	1.37	2.56
Mexico	MEX	18.61	4.94	2.25	74.2	27.7	1.39	3.35